ISGSR2007 First International Symposium on Geotechnical Safety & Risk
Oct. 18~19, 2007 Shanghai
Tongji University, China

Validation and Application of Reliability Analysis of Soil Slopes Using a Global Optimization Method

Jianfeng Xue
Former PhD student, University College Dublin, Dublin, Ireland, currently Geotechnical Engineer, Coffey Geotechnics Pty Ltd, Perth, Australia

Kenneth Gavin
University College Dublin, ,Dublin, Ireland

ABSTRACT: The validation and application of a new method for analyzing slope stability problems, which simultaneously locates the critical slip surface and the reliability index of a slope, is presented in this paper. The method allows both circular and non-circular slip surfaces to be analysed. The soil properties are considered as random variables and translated into polar coordinates, with which the reliability index of the slope can be formulated as a function of soil properties and the slip surface. The determination of the reliability index of a slope is solved using a global optimization technique. The validation proves the method to be computationally more efficient than a Monte Carlo simulation. When applied to the back-analysis of a railway embankment slope failure, where traditional deterministic approaches return factors of safety greater than unity, it successfully predicted the shape of the failure surface, and indicated poor overall reliability.

1 INTRODUCTION

Deterministic methods are widely used in slope stability analyses, although, they are severely limited in that the variability of soil properties cannot be incorporated into the analysis in a consistent manner. It is therefore impossible to quantify how conservative, or alternatively unsafe, a given FOS might be. This is in fact determined in a non-transparent way when the designer chooses the soil parameters for use in the design.

A number of workers have used reliability theory in slope stability analysis. However, due largely to mathematical complexities, the application have been partial, for example to examine the effect of varying soil properties on the reliability index of a fixed slip surface, located using a deterministic approach. To overcome such limitations a new method of analyzing slope stability problems, which simultaneously locates the critical slip surface, and the minimum reliability index of a slope, was developed by Xue and Gavin (2007). In this approach the soil properties are considered as random variables and are (for mathematical convenience) translated into polar coordinates, with which the reliability index of the slope can be formulated as a function of the soil properties and the geometry of the slip surface. Brief details of the method are presented. Comparing its performance with a full Monte-Carlo simulation of a benchmark slope validates the method. The model used in the validation is a development of the original in that it allows both circular and non-circular slip to be analyzed. In the final part of the paper the performance of the method in predicting the likely failure of a partly saturated, railway embankment slope is compared to a deterministic analysis.

2 LIMIT STATE FUNCTION

In reliability theory, the performance of a system is evaluated through the assessment of the probability of failure, which is defined as the probability that the demand will equal, or exceed the capacity:

$$P_f = P(g(X) = (C - D) \le 0) \quad [1]$$

in which P_f is the probability of failure, $g(X)$ is the limit state function, C is the capacity of a structure and D is the demand. It can be calculated by:

$$P_f = \int_{g(X) \leq 0} f(X) dX \qquad [2]$$

where $f(X)$ is the density function of random variables.

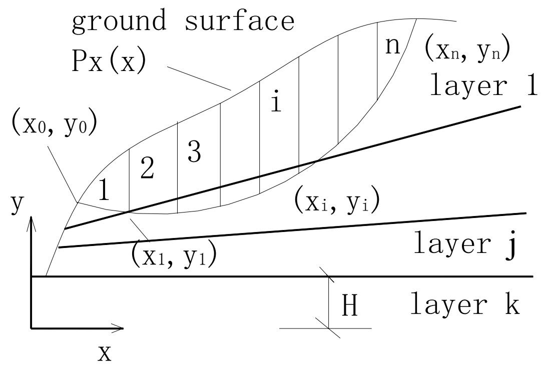

Fig.1 Layered slope with n slices in Bishop method.

For soil slopes, the limit state function can be expressed as:

$$g(x) = FOS - 1 \qquad [3]$$

where FOS is the Factor of Safety determined (for example) using the limit equilibrium method. Adopting the simplified Bishops method (Chowdhury 1978), and assuming the failure surface shown in Fig.1, the Factor of Safety can be expressed as:

$$FOS = \frac{\sum_{i=1}^{n} \left[c_i \Delta x_i + (W_i - u_i \Delta x_i) \tan \phi_i \right] \left(\frac{\sec \alpha_i}{1 + \tan \alpha_i \tan \phi_i / FOS} \right)}{\sum_{i=1}^{n} W_i \sin \alpha_i} \qquad [4]$$

in which W_i is the weight slice, α_i is the tangential angle of the base of the slice, Δx_i is the slice width, c_i is the cohesion of the soil on the base of the slice, u_i is the pore water pressure at the base of the slice, and ϕ_i is the friction angle of the soil at the base of the slice. At the limit state $FOS=1$, and substituting equation [4] into [3] we have:

$$g(x) = \frac{\sum_{i=1}^{n} \left[c_i \Delta x_i + (W_i - u_i \Delta x_i) \tan \phi_i \right] \left(\frac{\sec \alpha_i}{1 + \tan \alpha_i \tan \phi_i} \right)}{\sum_{i=1}^{n} W_i \sin \alpha_i} - 1 \qquad [5]$$

Assuming values of cohesion (c) and friction angle (ϕ) to be normally distributed random variables, the probability of failure of a slope can be obtained by integrating equation [2]. However, direct integration of Equation [2] is normally impracticable due to the nonlinearity of equation [5] and difficulties in accessing the density distribution function. For these reasons reliability index is often used in probabilistic approaches.

3 RELIABILITY INDEX

The Reliability index (β) can be described (with reference to Fig.2) as the distance between the origin and the mean value of the limit state function (*E[g(X)]*) in units of standard deviation (*σ[g(X)]*). By definition, the reliability index can be expressed as:

$$\beta = \frac{E[g(X)]}{\sigma[g(X)]} \qquad [6]$$

in which *E[g(X)]* and *σ[g(X)]* are the mean value and standard deviation of *g(X)*. This equation is normally used to evaluate the reliability of a design point *X*. The reliability index is often evaluated using the First Order Second Moment (FOSM) approach, which is based on approximating the limit state function using a Taylor's series expansion, see Harr (1987). However, the FOSM method has several disadvantages, most critically its lack of invariance for non-linear performance function, Low (1996).

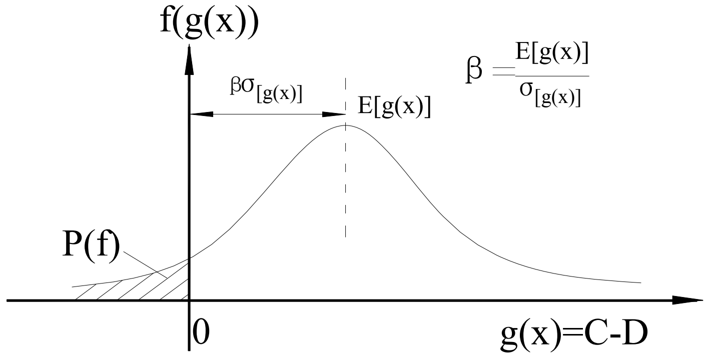

Fig.2 Relationship between reliability index and probability of failure

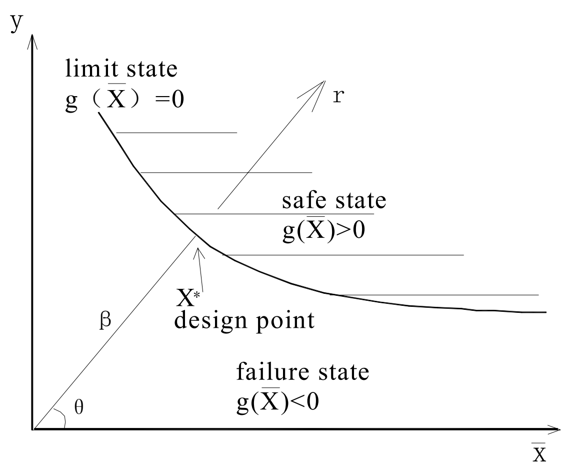

Fig.3 Relationship between reliability index and probability of failure

An invariant method to calculate the reliability index was proposed by Hasofer and Lind (1974) in which the random variables *[X]* can be standardized as reduced variables with their mean value and standard variation:

$$\overline{X_i} = \frac{X_i - E[X_i]}{\sigma[X_i]} \qquad \text{(i=1,2,.....,n)} \qquad [7]$$

where X_i is the i^{th} vector of variable space *[X]*. So in reduced variable space, the limit state function can be rewritten as:

$$g(X) = g(\overline{X_1}, \overline{X_2}, \cdots \overline{X_i}, \cdots, \overline{X_n}) \qquad [8]$$

And the reliability index can be expressed as the minimum distance from the origin to the limit state surface (see Fig.3). The point on the failure surface corresponding to β is called the design point or the most probable failure point. It can be described as:

$$\beta = \min_{\overline{X} \in \Psi} \left\{ (\overline{X})^T \overline{X} \right\}^{1/2} \qquad [9]$$

Where Ψ is the failure region.

4 RELIABILITY MODEL OF SLOPES

A model to simultaneously find the minimum reliability index and the critical probabilistic slip surface of an earth slope developed by Xue and Gavin (2007) is briefly described in this section. The model described herein differs from the original one in that the limit state function is defined with the simplified Bishop Method, which uses a circular slip surface.

For mathematical convenience in formulating the objective function, the distance from the design point to the origin of the reduced variable space is expressed in polar coordinates using the radial distance (r) and polar angle (θ) as x = r cos θ, and y = r sin θ, which can be extended to cover a large number of variables (N):

$$\begin{aligned}
\bar{x}_i &= r \cos\theta_1 \quad \cos\theta_2 \ldots\ldots \cos\theta_{N-2} \cos\theta_{N-1} \\
\bar{x}_{N-1} &= r \cos\theta_1 \sin\theta_2 \\
x_N &= \quad r \sin\theta_1
\end{aligned}$$

In which $(\overline{x}_1, \overline{x}_2, \ldots, \overline{x}_N)$ are rectangular coordinates, whilst, $(r, \theta_1, \theta_2, \ldots, \theta_{N-1})$ are the polar coordinates where:

$$\begin{gathered}
(-1/2)\pi \le \theta_i \le (1/2)\pi \\
(i = 1, 2, \cdots N-2), \\
0 \le \theta_{N-1} \le 2\pi .
\end{gathered}$$

Setting:

$$\begin{aligned}
&\cos\theta_1 \cos\theta_2 \cdots \cos\theta_{n-2} \cos\theta_{n-1} = \omega_1, \\
&\cos\theta_1 \cos\theta_2 \cdots \cos\theta_{n-2} \sin\theta_{n-1} = \omega_2, \\
&\cos\theta_1 \cos\theta_2 \cdots \sin\theta_{n-2} = \omega_3, \\
&\cdots \\
&\cos\theta_1 \sin\theta_2 = \omega_{n-1}, \\
&\sin\theta_1 = \omega_n
\end{aligned}$$

we can get a range of design value of these variables by considering their mean and standard deviation.

$$
\begin{aligned}
c_l &= E(c_l) - r\omega_l \sigma(c_l) \\
&= E(c_l) - r\bar{\sigma}(c_l) \qquad\qquad l = 1,2,\cdots,j \\
\tan(\phi_m) &= \tan(E[\phi_m]) - r\omega_{j+m}[\sec^2(E[\phi_m])]\sigma(\phi_m) \\
&= \tan(E[\phi_m]) - r[\sec^2(E[\phi_m])]\bar{\sigma}(\phi_m) \qquad m = 1,2,\ldots\ldots, j
\end{aligned} \qquad [10]
$$

Substituting equation [10] into [5] we can have:

$$
r = \frac{\sum_{i=1}^{n} \left\{[E(c_i)]\Delta x_i + (W_i - u_i \Delta x_i)[E(\tan\phi_i)]\right\} \frac{\sec\alpha_i}{1+\tan\varphi_i \tan\alpha_i} - \sum_{i=1}^{n}(W_i \tan\alpha_i)}{\sum_{i=1}^{n} \{\bar{\sigma}(c_i)\Delta x_i + (W_i - U_i)[\sec^2(\mathrm{E}[\phi_i])]\bar{\sigma}(\phi_i)\} \frac{\sec\alpha_i}{1+\tan\phi_i \tan\alpha_i}} \qquad [11]
$$

Noting that $\tan\phi_i$ on the right hand side of the equation is a function of r as shown in equation [10], a solution for r can be only be found by iteration.

The variables in Equation [11] describe the soil properties ($\theta_1, \theta_2, \ldots, \theta_{2k-1}$), where k is the number of layer in the slope, and the shape of the slip surface, ($x_0, y_1, y_2, \ldots, y_{n-1}, x_n$), where n is the number of slices in Bishop method. Combining these the reliability index β can be expressed as a function of $X(\theta_1, \theta_2, \ldots, \theta_{2k-1}, x_0, y_1, y_2, \ldots, y_{n-1}, x_n)$, and for a given vector X, the reliability index β and the critical slip surface can be obtained simultaneously. By choosing variables $X(\theta_1, \theta_2, \ldots, \theta_{2k-1}, x_0, y_1, y_2, \ldots, y_{n-1}, x_n)$ we can identify the minimum reliability index. This is a constrained nonlinear programming problem. It can be expressed numerically as:

Minimize: $r(X)$

$$
X = X(\theta_1, \theta_2, \cdots, \theta_{2k-1}, x_0, y_1, y_2, \cdots, y_{n-1}, x_n)
$$

Subject to: $X \in \Omega$, where Ω is the variable space. For

$$
(-1/2)\pi \le \theta_i \le (1/2)\pi \quad (i = 1,2,\cdots,2k-2),
$$

$$
0 \le \theta_{2k-1} \le 2\pi .
$$

For any slip surface, we have the following constraints:

(i)The slip surface must remain above the lower boundary of the region we define for the analysis. For example, for a search region with a lower boundary of $y \ge H$ as shown in Fig.1, we have:

$$
y_i \ge H \ , \quad (i = 0,1,2,\cdots,n) .
$$

(ii)The term ($1+\tan\phi_i\tan\alpha_i$) can become zero or negative near the toe of a steeply inclined slip surface when α_i has large negative values and $\tan\phi_i$ is non-zero. Therefore, as recommended by Whitman and Bailey (1967):

$$
\frac{\sec\alpha_i}{1+\tan\phi_i \tan\alpha_i} \ge 0.2 \qquad\qquad (i = 0,1,2,\cdots,n-1)
$$

should be satisfied.

(iii)All soil properties in equation [10], should be positive, so we have:

$$E(c_l) - r\bar{\sigma}(c_l) \geq 0 \qquad l = 1, 2, \cdots, j$$

$$\tan(E[\phi_m]) - r[\sec^2(E[\phi_m])]\bar{\sigma}(\phi_m) \geq 0 \qquad m = 1, 2, \ldots\ldots, j$$

The final constraint arises because of the assumption that the variables, which describe the soil parameters, are normally distributed. A revised version of the model under development uses log-normally distributed variables to eliminate this requirement.

The non-linear programming problem described in this paper is solved using a program a program based on the Genetic Algorithm (GA) method. Details of the method can be found in Goh (1999), Xue (2002) and Zolfaghari et al. (2005) amongst others. The program called Genetic Algorithm for Slope Stability Analysis (GASSA) was written in Visual C++. The population size used in the study was 100. The method incorporated a crossover point of 0.8 and a mutation rate of 0.06.

5 VALIDATION BY COMPARISON WITH MONT-CARLO SIMULATION

Monte-Carlo simulation (MCSM) is a powerful tool in probabilistic analysis. This population based method functions by randomly generating a number of variables according to their distribution. By evaluating the performance of the system under each set of these variables, the reliability of a system can be evaluated. It has been applied by Griffiths and Fenton (2004), Greco (1996) and Malkawi et al. (2001) in slope stability analysis. However, the accuracy of the method is proportional to the square root of the number of iterations, which results in greatly increased cost when improvements in accuracy are required, Baecher and Christian (2003).

In order to rigorously validate the new model a comparative analysis was performed where the reliability of a slope was evaluated using both GASSA and the MCSM method. The slope was a simple 5m high embankment with a 1:2 side slope. The mean soil parameters adopted were c = 7 kPa, ϕ = 10° and γ = 17.63 kN/m^3. The COVs adopted were 0.1 for cohesion and 0.05 for friction angle. A simplified Bishop's circular slip surface method was employed in both GASSA and the MCSM simulations. A total of 1000 variables were assigned for each soil property (cohesion and friction angle) in the MCSM analysis. The factor of safety for each group of soil properties was assessed using the GA method, giving 1000 realisations of FOS. The probability of failure was obtained by counting the number of events in which FOS < 1 was predicted.

The failure surfaces, reliability index and probability of failure from the MCSM simulation are shown in Fig.4. In total 19 slip surface with FOS < 1 were located, giving a probability of failure of 0.019 (1.9% of the slopes analysed), and a reliability index of 2.07. Each iteration of the analysis took about 1 second, giving a total run time of 1000 seconds. Obviously this would increase if improved accuracy were required, for example if in a given problem, a probability of failure of 0.1% was anticipated, at least 10,000 iterations would be required.

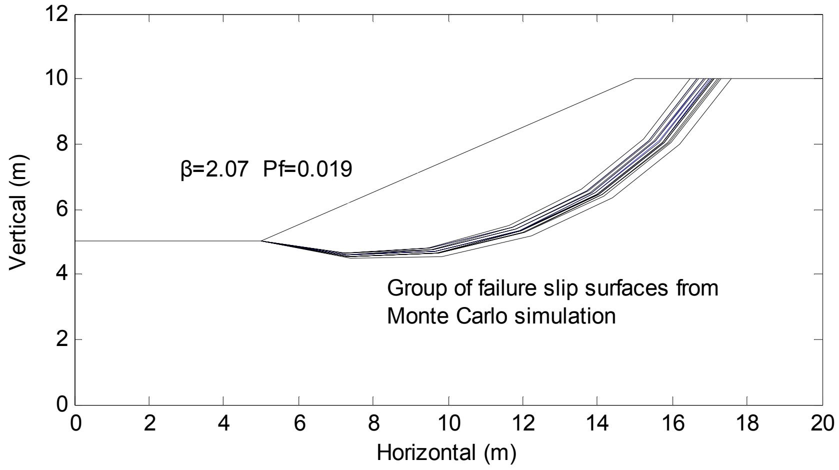

Fig.4 Group of failure surfaces from Monte Carlo simulation

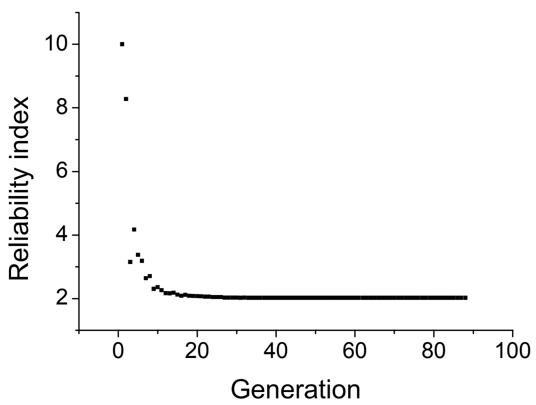

Fig.5 Convergence of reliability index with generations in GASSA.

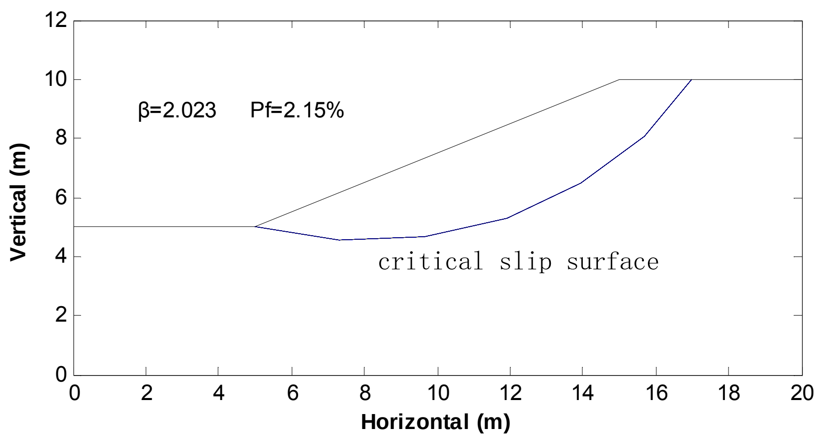

Fig.6 Critical slip surfaces from GASSA

When the slope was analysed using the GASSA method, the analysis took less than 10 seconds to complete. The β value calculated for each iteration is shown in Fig.6, which shows the method rapidly converges after about 20 iterations, whilst the termination criteria (COV of β < 0.001) was achieved after 88 iterations. The failure surface produced by GASSA is shown in Fig.6, where it is seen to be similar to that produced by the MCSM simulation. The probability of failure predicted by GASSA of 0.0215 or (2.15%) is similar to, yet higher than the MCSM simulation, suggesting that 1000 iterations of the MCSM method may not have been adequate to find the minimum β value. The reliability index of 2.023 predicted by GASSA is therefore slightly lower than the value of 2.07 predicted using the MCSM method.

6 BACK ANALYSIS OF OOLA EMBANKMENT

A survey of 150 year old man made embankments along the Irish railway network (Gavin et al 2006), revealed that the average embankment height and sloe angle were 4.5m and 45° respectively. Oola bank is a 4.25m high embankment, located near Limerick junction in the southern midlands of Ireland. With a slope angle of 45° it can be described as a typical Irish railway embankment. The slope has suffered two full-height failures in recent years. The most recent failure occurred in November 2000, where following a period of heavy rainfall, a 1m deep planar failure surface developed in the embankment slope (See Fig.7).

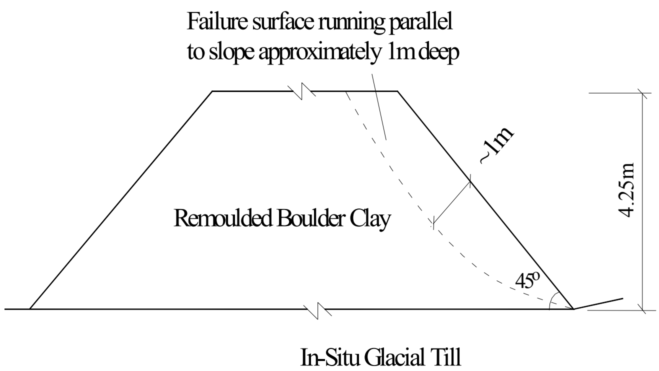

Fig.7 Shape of failure surface at Oola Embankment

A site investigation consisting of shell and auger boreholes revealed the sub-soil to be stiff, sandy gravelly Clay (boulder clay), with the water table level being 1m below original ground level. The embankment fill was remoulded boulder clay. Standard Penetration Test N values in the fill and subsoil were in the range 35-44. Particle size distribution curves show straight line grading typical of poorly sorted glacially deposited materials, with gravel and clay contents of 30 and 15% respectively. The boulder clay has a high fines content, with the percentage of material passing the 425μm being 55%, clearly classifying the material as a fine-grained soil. The soil has a very low clay mineral content, with the fines being composed of rock flour. Triaxial compression tests on boulder clay (Lehane and Faulkner 1998), show the material exhibits peak friction (ϕ_p) and constant volume friction angles (ϕ_{cv}) of 37° and 32°, with c′=0 and no tendency to form a residual shear surface.

Because of the high slope angle at which many natural and man-made boulder clay slopes stand, with slope angles significantly higher than ϕ_p being common, conventional design practice used to assess the stability of slopes assumes a nominal value of total cohesion, typically as high as 5-10 kPa to exist. Given the clay mineralogy of the parent material the mechanism controlling the high slope angles in the boulder clay is the presence of near surface suctions providing additional soil strength. Gavin et al. (2006) present a back-analysis of the failure at Oola bank in order to investigate the role of suction in supporting the steep initial slope, and to describe the failure mechanism, which developed in the slope. By varying the wetting front depth to match the observed slip surface, they concluded that a limiting matric suction of 3 kPa was necessary to provide an FOS of unity for the slope. This matric suction provided a contribution to total cohesion of between 1 and 2 kPa in the analysis. Using this approach a rainfall induced failure could only be predicted by arbitrarily reducing the operational c value in the deterministic analysis. Assuming a ϕ_p of 37°, and a c value in the range 1 – 3 kPa, the FOS predicted using a deterministic approach reduces from 1.337 (c = 3 kPa) to a minimum of 1.05 (c = 1 kPa). In effective stress analyses, FOS > 1.3 is deemed to be acceptable. It is apparent that the deterministic analysis does not provide a framework for rational analysis of failures of this type.

As a method of comparison the slope was analysed using GASSA. The same soil properties considered in the deterministic approach were adopted. The peak friction angle, the value of which can be assigned with some confidence, was given a COV of 0.1. The COV of the c value, which is relatively uncertain and contributes significantly to the failure mechanism, was varied from 0.1 to 0.6. The failure surface predicted using GASSA is in Fig.8, to be consistent with the actual failure surface. The results of both analyses, shown in Table 1, are best compared in terms of the classification of reliability using the US army classification system, USACE (1999). Adopting c values which are much lower than those used in routine design resulted in FOS values > 1 in the deterministic analysis and therefore did not predict slope failure. The results of the reliability analyses for the range of assumed soil properties, show the slopes are classified as having below average reliability ($\beta \leq 2.5$, $P_f \geq 0.006$), with most being classified as poor ($\beta \leq 2$, $P_f \geq 0.023$). The probabilistic analysis therefore allows the designer to adopt reasonable values for soil parameters, quantify their uncertainty, and produce a rational assessment of the probability of failure.

Table 1 Results from the analysis of Oola Embankment.

Case	Soil properties			Probabilistic method		Deterministic method
		c (kPa)	φ (˚)	β	P_f	FOS
A	mean	1	37			1.05
	COV	0.1	0.1	1.3	0.097	
B	mean	2	37			1.206
	COV	0.1	0.1	1.807	0.035	
		0.4	0.1	1.64	0.051	
		0.6	0.1	1.25	0.106	
C	mean	3	37			1.337
	COV	0.1	0.1	2.12	0.017	
		0.4	0.1	1.84	0.033	
		0.6	0.1	1.36	0.087	

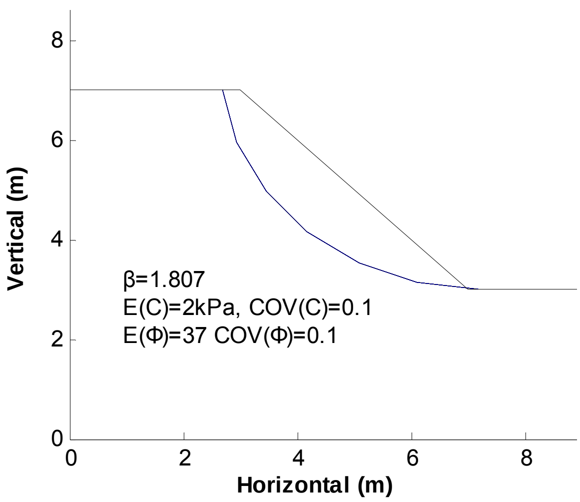

Fig.8 A critical probabilistic slip surface of Oola embankment predicted with GASSA.

7 CONCLUSION

The paper has presented a new method for determining the reliability index of a soil slope. The key feature of the method is the simultaneous determination of the reliability index and the critical slip surface. The method is an extension of the original model presented by Xue and Gavin (2007) in that both circular and non-circular slip surface approaches are incorporated. Comparing a Monte Carlo simulation of a benchmark slope, with the results obtained using the new model (GASSA), performed a comprehensive evaluation of the method. Both methods were seen to predict similar reliability indices, with GASSA being computationally much more efficient, with a run-time of 10 seconds, compared to 1,000 seconds for the Monte Carlo simulation. A further modification of the method to incorporate log-normally distributed variables is currently under development.

When GASSA was applied to the common design problem of assessing the stability of a compacted clay embankment, it provided an excellent prediction of the actual failure surface, which developed in the slope. More importantly, it highlighted the poor reliability of the slope using the same soil parameters, which in a deterministic analysis of the slope did not indicate incipient failure.

ACKNOWLEDGEMENT

This research is funded by Iarnród Éireann. The first author was the recipient of a Geotechnical Trust Fund award from the Geotechnical Society of Ireland. The paper was largely written whilst the second author was working as a research associate at the Urban Institute at University College Dublin. Financial assistance provided by the institute is gratefully acknowledged. The authors wish to thank Professor K.K. Phoon for his advice and guidance on this topic.

REFERENCES:

Baecher, G.B. and Christian, J.T. (2003) Reliability and statistics in geotechnical engineering. John Wiley & Sons Ltd.

Chowdhury, R.N. (1978) Slope analysis. Elsevler Science Publication Company.

Goh, A.T.C. (1999). Genetic algorithm search for critical slip surface in multiple-wedge stability analysis. Canadian Geotechnical Journal, 36(382-391).

Gavin, K.G., Xue, J.F, and Jennings, P. Assessment of the effect of pore pressures on the behaviour of railway foundations, Proceedings of XIIth Danube-European Conference on Geotechnical Engineering May 2006.

Greco, V.R. (1996) Efficient Monte Carlo technique for locating critical slip surface. Journal of Geotechnical Engineering, ASCE, 122(7): 517-525.

Griffiths, D.V. and Fenton, G.A. (2004) Probabilistic slope stability analysis by finite elements. Journal of Geotechnical and Geoenvironmental Engineering, 130(5): 507-518.

Harr, M.E. (1987). Reliability-based design in civil engineering. McGraw-Hill Book Company, New York.

Hasofer, A.M. and Lind, N.C. (1974) Exact and invariant Second-Moment code format. Journal of the Engineering Mechanics Division, ASCE, 100: 111-121.

LEHANE B.M. and FAULKNER A. Stiffness and strength characteristics of a hard lodgement till. Proceedings of 2nd International Symposium on the Geotechnics of Hard Soils and Soft Rocks, Naples, 1998, ***2****, 637-646.*

Low, B.K. (1996) Practical probabilistic approach using spreadsheet, Geotechnical Special Publication No. 58, Proceedings, Uncertainty in the Geologic Environment: From Theory to Practice. ASCE, Madison, Wisconsin, pp. 1284-1302.

Malkawi, A.H., Hassan, W.F. and Sarma, S.K. (2001) Global search method for locating general slip surface using Monte-Carlo techniques. Journal of Geotechnical and Geoenvironmental Engineering, 127(08): 688-698.

USACE (1999) ETL 1110-2-556, Risk-based analysis in geotechnical engineering for support of planning studies, Appendix A, page A1- A23. U.S. Army Corps Engineers Document.

Whitman, R.V. and Bailey, W.A. (1967) Use of computer for slope stability analysis. Journal of Soil Mechanics and Foundation. Div. ASCE, 93(4): 475-498.

Xue, J.F. (2002) Stability analysis with genetic algorithm and reliability study of earth slopes under earthquake. M.Eng Thesis Thesis, HuaQiao University, Quanzhou China.

Xue, J.F. and Gavin, K. (2007) Simultaneous determination of critical slip surface and minimum reliability index for earth slopes. International Journal of Geotechnical and Geoenvironmental (ASCE), 133(7): 878-886.

Zolfaghari, A.R., Heath, A.C. and McCombie, P.F. (2005) Simple genetic algorithm search for critical non-circular failure surface in slope stability analysis. Computers and Geotechnics, 32: 139-152.